Math 522 Exam 11 Solutions

1. Is 99 a quadratic residue modulo 151? Be sure to indicate in any symbols used whether they are Legendre or Jacobi symbols.
   BONUS: Is 151 a quadratic residue modulo 99?

   151 is prime, hence we can work entirely with Legendre symbols. $\left(\frac{99}{151}\right) = \left(\frac{3}{151}\right)^2\left(\frac{11}{151}\right) = \left(\frac{11}{151}\right)$. Now $11 \equiv 151 \equiv 3 \pmod 4$, so by quadratic reciprocity this equals $-\left(\frac{151}{11}\right) = -\left(\frac{8}{11}\right) = -\left(\frac{2}{11}\right)^3 = -\left(\frac{2}{11}\right) = -(-1)^{\frac{11^2-1}{8}} = -(-1)^{15} = 1$. Hence 99 is a quadratic residue modulo 151.

   99, however, is not prime, so we must work with Jacobi symbols. $\left(\frac{151}{99}\right) = \left(\frac{52}{99}\right) = \left(\frac{2}{99}\right)^2\left(\frac{13}{99}\right) = \left(\frac{13}{99}\right)$. Because $13 \equiv 1 \pmod 4$, by quadratic reciprocity this equals $\left(\frac{99}{13}\right) = \left(\frac{8}{13}\right) = \left(\frac{2}{13}\right)^3 = \left(\frac{2}{13}\right) = (-1)^{\frac{13^2-1}{8}} = (-1)^{21} = -1$. Had the answer been 1, this would be inconclusive; however since the answer is -1 we can conclude that 151 is not a quadratic residue modulo 99.

2. For all odd primes p, prove that

$$\left(\frac{5}{p}\right) = \begin{cases} 1 & \text{if } p \equiv 1, 9, 11, 19 \pmod{20} \\ -1 & \text{if } p \equiv 3, 7, 13, 17 \pmod{20} \\ 0 & \text{if } p = 5 \end{cases}$$

   By the division algorithm, we can set $p = 20k + a$, for some $0 \le a < 20$. Since p is an odd prime, in fact $a \in \{1, 3, 5, 7, 9, 11, 13, 17, 19\}$. If $a = 5$, then $5|p$ so in fact $p = 5$. Otherwise, since $5 \equiv 1 \pmod 4$, by quadratic reciprocity $\left(\frac{5}{p}\right) = \left(\frac{20k+a}{5}\right) = \left(\frac{a}{5}\right)$. For convenience, we calculate this for a complete residue system $\{1, 2, 3, 4\}$. $\left(\frac{1}{5}\right) = 1$, $\left(\frac{2}{5}\right) = (-1)^{\frac{5^2-1}{8}} = (-1)^3 = -1$, $\left(\frac{3}{5}\right) = \left(\frac{5}{3}\right) = \left(\frac{2}{3}\right) = (-1)^{\frac{3^2-1}{8}} = -1$, $\left(\frac{4}{5}\right) = \left(\frac{2}{5}\right)^2 = 1$.

   Putting it all together, we have $\left(\frac{1}{5}\right) = \left(\frac{11}{5}\right) = 1$, $\left(\frac{3}{5}\right) = \left(\frac{13}{5}\right) = -1$, $\left(\frac{7}{5}\right) = \left(\frac{17}{5}\right) = \left(\frac{2}{5}\right) = -1$, $\left(\frac{9}{5}\right) = \left(\frac{19}{5}\right) = \left(\frac{4}{5}\right) = 1$.

3. High score=103, Median score=85.5, Low score=55